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Detumbling and nutation canceling maneuvers with complete analytic reduction for axially symmetric spacecraft

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ABSTRACT

A new method is introduced to control and analyze the rotational motion of an axially symmetric rigid-body spacecraft. In particular, this motion is seen as the combination of the rotation of a *virtual sphere* with respect to the inertial frame, and the rotation of the body, about its symmetry axis, with respect to this sphere. Two new exact solutions are introduced for the motion of axially symmetric rigid bodies subjected to a constant external torque in the following cases: (1) torque parallel to the angular momentum and (2) torque parallel to the vectorial component of the angular momentum on the plane perpendicular to the symmetry axis. By building upon these results, two rotational maneuvers are proposed for axially symmetric spacecraft: a detumbling maneuver and a nutation canceling maneuver. The two maneuvers are the minimum time maneuvers for spherically constrained maximum torque. These maneuvers are simple and elegant, as they reduce the control of the three degrees-of-freedom nonlinear rotational motion to a single degree-of-freedom linear problem. Furthermore, the complete (both for the dynamics and for the kinematics) and exact analytic solutions are found for the two maneuvers. An extended survey is reported in the introduction of the paper of the few cases where the rotation of a rigid body is fully reduced to an exact analytic solution in closed form.

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1. Introduction

Spacecraft detumbling maneuvers and spacecraft nutation canceling maneuvers have critical importance and are often performed in astrodynamic applications. The detumbling maneuver is typically performed by most spacecraft after their separation from the rocket launcher, once the orbital flight state has been reached. This maneuver cancels any residual angular velocity and prepare the spacecraft for its nominal rotational motion condition. On the other hand, the nutation canceling maneuver is typically needed for spinning stabilized spacecraft whenever a transversal (with respect to the spinning axis) component of the angular velocity builds up as an effect of either external disturbance or reorientation control torque. This maneuver cancels the transversal components of the angular velocity, which causes the associated characteristic coning motion, and brings the spacecraft back to its nominal pure spinning condition about the symmetry axis.

The results presented in this paper apply to axially symmetric spacecraft, which constitute a large sub-class of all spacecraft. In particular, virtually all of the spinning stabilized spacecraft are axially symmetric.

While a vast literature exists regarding spacecraft rotational maneuvers, (see, for instance, [1,2]) no exact analytic reduction is generally found for those maneuvers.

The analysis of spacecraft rotational maneuvers is a practical application of the theory of the rotational motion of a rigid body.

The problem of the rotational motion of a rigid body can be divided into two parts. The *dynamic problem* aims to

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obtain the angular velocity of the body with respect to an inertial reference by starting from the knowledge of the initial angular velocity and the history of the applied torque. On the other hand, the *kinematic problem* focuses on determining the current orientation of the body from the knowledge of the initial orientation and the history of the angular velocity.

The exact analytic solution for the rotational motion of a rigid body exists only for a small number of special cases, which are surveyed here below.

When no external torque acts on the rigid body (Euler–Poinsot case) the rotational motion of a generic triaxial body has an exact analytic solution for both the dynamic and kinematic problems [3–8].

When only the torque due to gravity acts on the rigid body, exact solutions exist for both the dynamic and kinematic problems only in the following two special cases:

1. Lagrange–Poisson heavy-top case: Rigid body under gravity force with two equal principal moments of inertia at the fixed point and the center of mass along the third axis of inertia [6,9–12].
2. Kovalevskaya heavy-top case: Rigid body under gravity force with two equal principal moments of inertia at the fixed point, value of the third moment of inertia equal to half of the value of the moment inertia about the other two axes, and the center of mass in the plane of equal moments of inertia [6,13].

When the rigid body is subjected to an external torque which is dependent at most on time (self-excited body), the complete (i.e. for both the kinematics and the dynamics) exact analytic solution exists, in a form totally reduced to elementary functions, only in the following cases:

1. Single degree-of-freedom rotation of a generic rigid body with the applied torque along one of the three principal axes of inertia and initial angular velocity along the same axis.
2. Rigid body with axially symmetric ellipsoid of inertia subjected to a torque proportional to the current angular momentum vector [14].
3. Rigid body with axially symmetric ellipsoid of inertia subjected to the viscous friction modeled as a torque equal to the opposite of the angular velocity vector left multiplied by a matrix of constant coefficients [14,15].
4. Rigid body with axially symmetric ellipsoid of inertia subjected to the superposition of a viscous friction torque directed as the angular momentum vector and a second torque, which is either constant and inertially fixed or constant and fixed with the axis of symmetry of the body [16].
5. Rigid body with spherical ellipsoid of inertia subjected to a constant torque fixed with the body and arbitrary initial angular velocity [17].
6. Rigid body with axially symmetric ellipsoid of inertia subjected to a constant torque parallel to the symmetry axis [18].
7. Rigid body with axially symmetric ellipsoid of inertia subjected to a torque constant in magnitude, perpendicular to the symmetry axis and fixed with the body, and initial angular velocity perpendicular to the symmetry axis [18].
8. Rigid body with axially symmetric ellipsoid of inertia subjected to a torque constant in magnitude, perpendicular to the symmetry axis and rotating at a specific constant rate about the symmetry axis [18].

No other exact analytic solutions exist besides the one listed above, to the best knowledge of the author. The limitation in the number of these solutions constitutes a critical knowledge gap in the field of rigid body mechanics, as previously noticed by Ivanova [14].

In order to find the solutions to the problems listed as items 2–4 above, Ivanova [14,15] utilize an original view of the turn-tensor (rotation matrix) as a function of right and left angular velocities vector, introduced by Ivanova [19], and introduce a theorem which is analogous to the reduction principle previously established by Hestenes [20].

In order to find the solution for the problem listed as item five above, Romano [17] exploits the parametrization of the rotation by three complex numbers in order to reduce the kinematic differential equation to an equation of Riccati which is then solved through appropriate choices of substitutions, thereby yielding a completely reduced analytic solution in terms of confluent hypergeometric functions. The parametrization of the rotation used by [17] was introduced by [21], and is very similar to one previously introduced by Darboux [22] (see also [23]). Before Romano [17] and Ivanova [24] derives a partially reduced solution for the same problem while studying the motion of a ball on a rough plate. This partial solution includes not reduced integrals spanning over the time duration of the maneuver.

In order to find the solutions for the problem listed as item six o eight above, Romano [18] exploits the Hestenes' reduction principle and the solution of Romano [17]. An incomplete solution for the problem in item six was previously sketched by Lurie [23]. In particular, Lurie [23] uses the Cayley–Klein's parameters in order to express the dynamics and kinematics equations as a Darboux differential equation problem, then transforms the Darboux problem into a Weber differential equation problem, which is nevertheless left unsolved.

Finally, a partially reduced analytic solution exists (limited for the dynamic problem) for the case of an axially symmetric body subjected to a constant torque [25]. In particular, Tsiotras and Longuski [25], building upon the development of Bödewadt [26], use a complex form expression of the Euler's dynamic equations and give a solution for the angular velocity involving a not reduced Fresnel integral which has to be evaluated over the duration of the maneuver.

Many researchers have proposed approximate solutions for the motion of a rigid body. For instance, as regards the kinematic problem, Iserles and Nørsett [27] study the solution in terms of series expansion for the more general problem of solving linear differential equation in Lie Groups, building upon the work of Magnus [28]. Celledoni and Saefstroem [29] propose ad hoc numerical integration algorithms. Finally, Livneh and Wie [30] introduce approximate results for the motion of a triaxial rigid body subjected to a constant torque.

This paper introduces for the first time, to the best knowledge of the author, the following original contributions:

1. A version which is simplified, without loss of generality, with respect to the one proposed in [17] is introduced for the exact analytic solution of the motion of a rigid body with spherical ellipsoid of inertia and subjected to a constant torque.
2. A new exact analytic solution is introduced for the motion of a rigid body with axially symmetric ellipsoid of inertia subjected to a torque constant in magnitude and parallel to the angular momentum vector.
3. A new exact analytic solutions is introduced for the motion of a rigid body with axially symmetric ellipsoid of inertia subjected to a torque constant in magnitude and parallel to the vectorial component of the angular momentum vector on the plane perpendicular to the axis of the body.
4. A detumbling maneuver with complete analytic reduction is introduced for axially symmetric spacecraft.
5. A nutation canceling maneuver with complete analytic reduction is introduced for axially symmetric spacecraft.

The results presented in this paper have both a theoretical importance for their basic mechanics implications and a practical importance for the astrodynamic applications related to spacecraft rotational maneuvering.

The paper is organized as follows: Section 2 introduces the dynamic and kinematic equations for the rotational motion of a generic rigid body, Section 3 introduces the simplified solution for the motion of a spherically symmetric body, Section 4 introduces the equation of motion for an axially symmetric rigid body, Section 5 presents the new exact analytic solutions, Section 6 introduces the detumbling and nutation canceling maneuvers. Finally, Section 7 concludes the paper.

2. Rotation of a triaxial rigid body

For a generic rigid body the Euler's dynamic equations of rotation are [31]

$$\begin{aligned} I_1\dot{p} &= (I_2-I_3)qr+m_1, \\ I_2\dot{q} &= (I_3-I_1)rp+m_2, \\ I_3\dot{r} &= (I_1-I_2)pq+m_3, \end{aligned} \tag{1}$$

where $\{I_i : i=1,2,3\}$ are the principal moments of inertia, $\{p,q,r\}$ are the components of the absolute angular velocity along the principal coordinate system B, and $\{m_i : i=1,2,3\}$ are the components of the resultant external torque.

The rotation matrix $R_{NB} \in SO(3)$ from the principal body fixed coordinate system B to the inertial coordinate system N obeys the following kinematic differential equation [6]

$$\dot{R}_{NB} = R_{NB}\Omega(^B\omega_{BN}), \tag{2}$$

where

$$\Omega(^B\omega_{BN}) = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}. \tag{3}$$

In general, Eqs. (1) and (2) do not have an exact analytic solution. When the matrix $\Omega(t)$ commutes with its time integral, Eq. (2) has the solution [32]

$$R_{NB}(t) = R_{NB}(0)\exp\left(\int_0^t \Omega(^B\omega_{BN}(\xi))\,d\xi\right), \tag{4}$$

where $\exp(\cdot)$ indicates the matrix exponential.

3. Exact analytic solutions for the motion of a rigid body with spherical ellipsoid of inertia

Let us assume, in this section, that the rigid body has a spherical ellipsoid of inertia, with

$$I_1 = I_2 = I_3 = I. \tag{5}$$

This section summarizes the results introduced by Romano [17] for the motion of a rigid body with spherical ellipsoid of inertia and subjected to a constant torque. In particular, the mathematical developments here presented are simplified with respect to Romano [17], without loosing generality, by choosing a particular body-fixed coordinate system such that one component of the initial angular velocity is zeroed.

The developments reported in this section will be used in the following sections of this paper in order to analyze the motion of an axially symmetric rigid body.

For a rigid body having a spherical ellipsoid of inertia, and subjected to a constant body-fixed torque, the kinematic differential equations in terms of the stereographic complex rotation variables w_k [21,33] are [17]

$$\dot{w}_k = \tfrac{1}{2}p_0 w_k^2 - i(r_0+Ut)w_k + \tfrac{1}{2}p_0, \quad k=1,2,3. \tag{6}$$

where i is the imaginary unit and, without loosing generality (see Corollary 1 here below), we assume the initial angular velocity and the acting torque (normalized by the value of the moment of inertia) to have components $\{p_0, 0, q_0\}$ and $\{0,0,U\}$, respectively, in the chosen body-fixed coordinate system B.

Theorem 1. *Given p_0, r_0, and U real numbers, the general solution for each one of Eq. (6), governing the rotational kinematics of a rigid body with spherical ellipsoid of inertia, initial angular velocity components $p(0)=p_0$, $q(0)=0$, and $r(0)=r_0$ along the three body fixed axes σ_1, σ_2, and σ_3 (constituting the coordinate system B), and subjected to a constant torque U, normalized by the value of the moment of inertia and directed along the axis σ_3, is the following:*

$$w(t,c) = \frac{(1+i)\sqrt{U}}{3p_0}[6z+G(z,c)], \tag{7}$$

with

$$G(z,c) := \frac{2\,{}_1F_1\left(\frac{3-\nu}{2},\frac{5}{2};z^2\right)(\nu-1)z^2 + 6\,{}_1F_1\left(1-\frac{\nu}{2},\frac{3}{2};z^2\right)c\nu z - 3\,{}_1F_1\left(\frac{1-\nu}{2},\frac{3}{2};z^2\right)}{{}_1F_1\left(-\frac{\nu}{2},\frac{1}{2};z^2\right)c + {}_1F_1\left(\frac{1-\nu}{2},\frac{3}{2};z^2\right)z}, \tag{8}$$

where ${}_1\mathrm{F}_1$ denotes the confluent hypergeometric function [34], $c \in C$ is the constant of integration and

$$z := \frac{(1+i)(r_0+Ut)}{2\sqrt{U}}, \quad \nu := -1 - \frac{ip_0^2}{4U}. \tag{9}$$

Corollary 1. *The solution in terms of the rotation matrix, which corresponds to the solution given by Theorem 1 in terms of stereographic rotation variables, is*

$$R_{NB}(t) = [r_{kj}(t)], \quad k,j = 1,2,3, \tag{10}$$

with

$$r_{k1} = \frac{i(w_k - \overline{w_k})}{1+|w_k|^2}, \quad r_{k2} = \frac{w_k + \overline{w_k}}{1+|w_k|^2},$$
$$r_{k3} = \frac{1-|w_k|^2}{1+|w_k|^2}, \quad k = 1,2,3, \tag{11}$$

where $w_k = w(t, c_k)$, being $w(t, c_k)$ given by Eq. (7) with the following values for the initial conditions (obtained by considering the rotation matrix R_{NB} to be equal to the identity matrix at the initial time $t = 0$, without loss of generality):

$$c_1 = -\frac{(1+i)}{6\sqrt{U}}\left\{\frac{{}_1\mathrm{F}_1\left(\frac{1-\nu}{2},\frac{3}{2};\frac{ir_0^2}{2U}\right)[6r_0^2+3p_0r_0+6iU]+2g}{2\,{}_1\mathrm{F}_1\left(1-\frac{\nu}{2},\frac{3}{2};\frac{ir_0^2}{2U}\right)\nu r_0 + {}_1\mathrm{F}_1\left(-\frac{\nu}{2},\frac{1}{2};\frac{ir_0^2}{2U}\right)(p_0+2r_0)}\right\}$$

$$c_2 = \frac{(1-i)}{6\sqrt{U}}\left\{\frac{{}_1\mathrm{F}_1\left(\frac{1-\nu}{2},\frac{3}{2};\frac{ir_0^2}{2U}\right)[6ir_0^2-3p_0r_0-6U]-2g}{2\,{}_1\mathrm{F}_1\left(1-\frac{\nu}{2},\frac{3}{2};\frac{ir_0^2}{2U}\right)\nu r_0 + {}_1\mathrm{F}_1\left(-\frac{\nu}{2},\frac{1}{2};\frac{ir_0^2}{2U}\right)(ip_0+2r_0)}\right\}$$

$$c_3 = \frac{(1+i)}{6r_0\sqrt{U}}\left\{\frac{g-3\,{}_1\mathrm{F}_1\left(\frac{1-\nu}{2},\frac{3}{2};\frac{ir_0^2}{2U}\right)(r_0^2+iU)}{{}_1\mathrm{F}_1\left(1-\frac{\nu}{2},\frac{3}{2};\frac{ir_0^2}{2U}\right)\nu + {}_1\mathrm{F}_1\left(-\frac{\nu}{2},\frac{1}{2};\frac{ir_0^2}{2U}\right)}\right\}, \tag{12}$$

where ν is defined as in Eq. (9), and

$$g := {}_1\mathrm{F}_1\left(\frac{3-\nu}{2},\frac{5}{2};\frac{ir_0^2}{2U}\right)(\nu-1)r_0^2. \tag{13}$$

In the case of having a general body-fixed coordinate system which is not coincident with a given inertial coordinate system N at the initial time, and whit respect to which the constant torque is not directed along the third axis and the initial angular velocity has non-zero components along the three axes, the results of Theorem 1 and Corollary 1 are still applicable, with the modification reported in the following corollary.

Corollary 2. *Let us consider that the direction of the normalized external torque $\underline{u} = \underline{m}/I$ is identified in a generic body-fixed coordinate system F by the unit vector*

$${}^F\hat{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{14}$$

and that the initial angular velocity $\underline{\omega}_0$ is expressed in F by ${}^F\omega_0$.

Finally, let us assume that the initial orientation of F with respect to the inertial coordinate system N is given by $R_{FN}(0)$. Then, by exploiting the property of successive rotations, it results

$$R_{FN}(t) = R_{FB}R_{BN}(t)R_{BF}R_{FN}(0), \tag{15}$$

where R_{FB} is a time-independent rotation matrix which describes the orientation of the generic body-fixed coordinate system F with respect to the particular body-fixed coordinate system B, in which the torque coincides with the third axis and the initial angular velocity has zero component along the second axis.

In particular, the matrix R_{FB} is

$$R_{FB} = [{}^F\sigma_1, {}^F\sigma_2, {}^F\sigma_3], \tag{16}$$

where $\{{}^F\sigma_i : i = 1,2,3\}$ are the column matrices of components in the coordinate system F of the unit vectors $\{\sigma_i : i = 1,2,3\}$ identifying the axes of the coordinate system B. These column matrices are given by

$${}^F\sigma_3 = {}^F\hat{u},$$

$${}^F\sigma_2 = \frac{({}^F\hat{u})^{\times}\,{}^F\omega_0}{|({}^F\hat{u})^{\times}\,{}^F\omega_0|},$$

$${}^F\sigma_1 = ({}^F\sigma_2{}^{\times})\,{}^F\sigma_3, \tag{17}$$

where $(\bullet)^{\times}$ indicates the matricial form of the vector product (skew symmetric matrix having the same structure as the matrix $\Omega({}^B\omega_{BN})$ in Eq. (3)). Consequently, it yields

$${}^Bu = R_{BF}{}^Fu = \begin{Bmatrix} 0 \\ 0 \\ U \end{Bmatrix}; \quad {}^B\omega_0 = R_{BF}{}^F\omega_0 = \begin{Bmatrix} p_0 \\ 0 \\ r_0 \end{Bmatrix}. \tag{18}$$

Finally, $R_{BN}(t)$ is obtained by transposing the resulting matrix of Eq. (10) of Corollary 1, and by using Eq. (18).

4. Rotation of an axially symmetric rigid body

Let us assume, for the rest of this paper, that the rigid body has a revolution ellipsoid of inertia, with

$$I_1 = I_2 = I \neq I_3. \tag{19}$$

The absolute angular momentum of the body ($\underline{h}$) can therefore be expressed as

$$\underline{h} = I\underline{\omega}_{\perp} + I_3\underline{\omega}_{\parallel}, \tag{20}$$

where $\underline{\omega}_{\perp}$ is the orthogonal vectorial component of the angular velocity $\underline{\omega}$ on the plane normal to the body axis $\underline{e}$ and $\underline{\omega}_{\parallel} = (\underline{\omega} \cdot \underline{e})\underline{e}$ is the orthogonal vectorial component of the angular velocity along the body axis.

Alternatively, by following the development of Hestenes [20], the absolute angular momentum of the body can be also expressed by

$$\underline{h} = I\underline{\omega} + (I_3 - I)(\underline{\omega} \cdot \underline{e})\underline{e}. \tag{21}$$

Conversely, the angular velocity can be seen, at any instant of time, as the sum of two vectorial components, whose one is parallel to the angular momentum vector ($\underline{\omega}_h$) and the other one is parallel to the symmetry axis ($\underline{\omega}_e$), i.e.

$$\underline{\omega} = \underline{\omega}_h + \underline{\omega}_e. \tag{22}$$

Notably, the vectorial components $\underline{\omega}_h$ and $\underline{\omega}_e$ are not orthogonal. Therefore, the magnitudes of $\underline{\omega}_h$ and $\underline{\omega}_e$ are not given by the scalar product of the vector $\underline{\omega}$ with the unit vectors $\underline{h}$ and $\underline{e}$. In other words, it yields $\underline{\omega}_e \neq \underline{\omega}_{\parallel}$. See also Fig. 1.

In particular, from Eqs. (21) and (22), by taking into account that $\underline{\omega} \cdot \underline{e} = (\underline{h} \cdot \underline{e})/I_3$, it yields

$$\underline{\omega}_h = \frac{\underline{h}}{I}, \quad \underline{\omega}_e = \frac{(I - I_3)}{I_3}\left(\frac{\underline{h}}{I} \cdot \underline{e}\right)\underline{e} = A(\underline{\omega}_h \cdot \underline{e})\underline{e}, \tag{23}$$

with A is a constant defined as $A = (I - I_3)/I_3$.

This is an interesting result: the vectorial component of the angular velocity along $\underline{e}$ is function of the vectorial angular velocity along $\underline{h}$. Therefore, if $\underline{\omega}_h$ is known, the full angular velocity of the axially symmetric rigid body is also known.

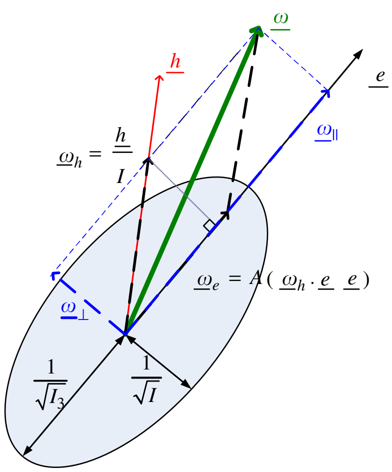

Fig. 1. Angular momentum, angular velocity and axis of symmetry for a generic axially symmetric body, represented by its inertia ellipsoid.

Furthermore, by inserting the first equality of Eq. (23) into the Euler's equation, it gives

$$\dot{\underline{h}} = I\dot{\underline{\omega}}_h = \underline{m}. \tag{24}$$

Eqs. (22) and (24) are the mathematical expression of the *Reduction Theorem* [20] which can be stated as follows: the evolution in time of the absolute angular momentum of an axially symmetric body subjected to an external torque $\underline{m}$ is equal to the evolution in time of the angular momentum of a "virtual" homogeneous spherical body (here doubled as *Virtual Sphere*) with the same value of the transversal inertia of the axially symmetric body and subjected to the same external torque.

In order to analyze in more details the motion of the axially symmetric body, we consider the following three Cartesian coordinate systems:

1. A principal coordinate system B attached to the axially symmetric body and centered at its center of mass.
2. A coordinate system S attached to the *Virtual Sphere* and centered at its center of mass.
3. An inertially fixed coordinate system N.

Assumptions 1. Without loosing generality we assume that the coordinate system B has its third axis parallel to the axis of inertial symmetry of the body ($\underline{e}$), and that the coordinate systems B and S have superimposed axes at the initial time ($t = 0$), i.e. that $R_{BS}(0)$ is an identity matrix.

Because of Eqs. (22) and (24) and Assumptions 1, we can see $\underline{\omega}_h$ as the absolute angular velocity of the coordinate system S with respect to N, and $\omega_e = A(\underline{\omega}_h \cdot \underline{e})$ as the angular rate characterizing the relative spinning of the coordinate system B with respect to the coordinate system S about their common third axis.

Consequently, we can rewrite Eq. (22) in the following matrix form

$${}^B\omega_{BN}(t) = R_{BS}(t){}^S\omega_{SN}(t) + {}^B\omega_{BS}(t), \tag{25}$$

where ${}^B\omega_{BN}(t)$ is the column vector of components of the angular velocity of B with respect to N resolved in B, and

$${}^S\omega_{SN}(t) = R_{SB}(t){}^B\omega_h = \frac{R_{SB}(t){}^Bh(t)}{I}; \quad {}^B\omega_{BS}(t) = {}^B\omega_e. \tag{26}$$

The kinematic solution associated to Eq. (25) is

$$R_{BN}(t) = R_{BS}(t)R_{SN}(t), \tag{27}$$

as it can be immediately demonstrated.

Furthermore, Eq. (24) can be rewritten in matrix form as

$${}^S\dot{\omega}_{SN} = \begin{Bmatrix} \dot{\overline{p}} \\ \dot{\overline{q}} \\ \dot{\overline{r}} \end{Bmatrix} = \frac{1}{I}{}^Sm = \frac{1}{I}R_{SB}{}^Bm, \tag{28}$$

with the initial condition

$${}^S\omega_{SN}(0) = \frac{R_{SB}(0){}^Bh_0}{I} = \begin{Bmatrix} p_0 \\ q_0 \\ \frac{I_3}{I}r_0 \end{Bmatrix}, \tag{29}$$

where (p, q, r) are the components of $\underline{\omega}$ along B and $(\overline{p}, \overline{q}, \overline{r})$ are the components of $\underline{\omega}_h$ along S.

In particular, for all of the cases when the external acting torque depends at most on time (self-excited rigid body), from Eqs. (28) and (23) it results

$$\omega_e(t) = A\bar{r}(t) = A\left(\frac{I_3}{I}r_0 + \frac{1}{I}\int_0^t m_e(\xi)\,d\xi\right), \tag{30}$$

where $m_e = \underline{m} \cdot \underline{e}$.

Finally, the evolution in time of R_{BS}, describing the elementary rotation of coordinate system B with respect to S about the axis $\underline{e}$, is given by (see also Eqs. (2) and (4))

$$R_{BS}(t) = R_{BS}(0)\exp\left(\int_0^t \Omega({}^S\omega_{SB})(\xi)d\xi\right) = \begin{bmatrix} \cos(f(t)) & \sin(f(t)) & 0 \\ -\sin(f(t)) & \cos(f(t)) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{31}$$

with

$$f(t) = \int_0^t \omega_e(\xi)\,d\xi \tag{32}$$

and

$${}^S\omega_{SB} = {}^B\omega_{SB} = \begin{Bmatrix} 0 \\ 0 \\ -\omega_e(t) \end{Bmatrix}. \tag{33}$$

In conclusion, Eqs. (25) and (27) determine the dynamics and the kinematics, respectively, of the motion of an axially symmetric rigid body (with moments of inertia I and I_3). In particular, the overall motion of the axially symmetric body (i.e. the motion of B w.r.t. N) is decomposed into the combination of the absolute rotation of an homogeneous *Virtual Sphere* with principal moment of inertia I (i.e. the motion of S w.r.t. N), and the relative spinning of the axially symmetric rigid body with respect to the *Virtual Sphere* about the inertial symmetry axis $\underline{e}$ (i.e. the motion of B w.r.t. S).

The critical advantage of the proposed approach is that the two composing motions can be solved independently in several significant cases, by using Eqs. (25)–(33), and determining the remaining unknown quantities. These unknown quantities are, namely, the angular velocity components $\bar{p}(t), \bar{q}(t)$, obtained by integrating Eq. (28), and the rotation matrix $R_{NS}(t)$, obtained by integrating the following kinematic differential equation

$$\dot{R}_{NS} = R_{NS}\Omega({}^S\omega_{SN}). \tag{34}$$

5. New exact analytic solutions for the motion of an axially symmetric rigid body

The exact analytic solution for the complete dynamic and kinematic problems of an axially symmetric rigid body were introduced by Romano [18] for several significant cases, by exploiting the Reduction Theorem approach. Those cases are listed in the introduction of this paper.

The following two sections introduce two new results with respect to [18]. This results will be exploited later in this paper for the analysis of the detumbling and nutation canceling maneuvers of an axially symmetric spacecraft.

5.1. *Case of constant torque along the angular momentum vector*

The result introduced by the following theorem regards the case of applied constant torque along the initial angular momentum vector of the axially symmetric body, and therefore also along the angular momentum vector at any time during the motion, since the direction of the angular momentum vector cannot change in this case. Furthermore, the direction of the angular momentum vector is also the direction of the angular velocity of the *Virtual Sphere*, because of Eq. (23). The development in this section will be exploited later in this paper for the analysis of the detumbling maneuver of an axially symmetric spacecraft.

Theorem 2. *Given p_0, q_0, r_0, and U real numbers, the solution of the dynamic problem of determining, at any time t, the absolute angular velocity of a rigid body having two equal principal moments of inertia about the principal body axes σ_1 and σ_2, initial angular velocity components $p(0) = p_0$, $q(0) = q_0$, and $r(0) = r_0$ along the three principal axes, and subjected to an external torque of constant magnitude (m) and always parallel to the angular momentum vector of the body (and, therefore, to the angular velocity of the Virtual Sphere), is the following*:

$${}^B\omega_{BN}(t) = \begin{Bmatrix} p_0\cos(f(t)) + q_0\sin(f(t)) \\ -p_0\sin(f(t)) + q_0\cos(f(t)) \\ r_0 \end{Bmatrix}\left[1 + \frac{Ut}{|{}^S\omega_{SN}(0)|}\right], \tag{35}$$

where $U = m/I$,

$$f(t) = \left(\frac{I - I_3}{I}\right)r_0\left[t + \frac{Ut^2}{2|{}^S\omega_{SN}(0)|}\right] \tag{36}$$

and

$$|{}^S\omega_{SN}(0)| = \sqrt{p_0^2 + q_0^2 + \left(\frac{I_3}{I}r_0\right)^2}. \tag{37}$$

Furthermore, the solution of the correspondent kinematic problem of determining, at any time t, the orientation of the body with respect to the inertial frame, i.e. of the coordinate system B with respect to N, is given by

$$R_{BN}(t) = R_{BS}(t)R_{SN}(t) \tag{38}$$

where $R_{BS}(t)$ is given by Eq. (31) with $f(t)$ as in Eq. (36), and

$$R_{NS}(t) = R_{NS}(0)\exp\left(\int_0^t \Omega({}^S\omega_{SN}(\xi))\,d\xi\right). \tag{39}$$

Proof. This theorem directly follows from the analytical developments of Section 4. In particular the angular velocity of the *Virtual Sphere* is given by

$${}^S\omega_{SN}(t) = \begin{Bmatrix} \bar{p}(t) \\ \bar{q}(t) \\ \bar{r}(t) \end{Bmatrix} = \begin{Bmatrix} p_0 \\ q_0 \\ \frac{I_3}{I}r_0 \end{Bmatrix}\left[1 + \frac{Ut}{|{}^S\omega_{SN}(0)|}\right], \tag{40}$$

as obtained by integrating Eq. (28), with initial conditions given by Eq. (29) and

$$\frac{1}{I}R_{SB}{}^{B}m = \frac{{}^{B}m}{I} = U\frac{{}^{S}\omega_{SN}(0)}{|{}^{S}\omega_{SN}(0)|}, \tag{41}$$

which corresponds to the theorem's hypothesis of having a constant external torque $\underline{m}$ parallel to the initial angular velocity of the sphere.

Moreover, Eq. (35) is found through the use of Eq. (25), with ${}^{S}\omega_{SN}(t)$ given by Eq. (40), and ${}^{B}\omega_{BS}$ given by Eq. (33) with

$$\omega_e = A\overline{r}(t) = \left(\frac{I-I_3}{I}\right)r_0\left[1+\frac{Ut}{|{}^{S}\omega_{SN}(0)|}\right]. \tag{42}$$

Finally, the rotation matrix $R_{NS}(t)$ (given by Eq. (39)) is obtained by applying the method of Eq. (4), which can be used here as the commutation condition holds, with the angular velocity components given by Eq. (40).

A perhaps simpler alternative to obtain $R_{NS}(t)$ is to use the Euler's Theorem of rotation (see, for instance, [31])

$$\begin{aligned} R_{SN}(t) = \{&\cos(\phi(t))\mathcal{I} + [1-\cos(\phi(t))]aa^T \\ &-\sin(\phi(t))\Omega(a)\}R_{SN}(0), \end{aligned} \tag{43}$$

where $\Omega(\cdot)$ is the skew-symmetric matrix function appearing also in Eq. (3), $\mathcal{I}$ indicates the three by three identity matrix, a is the axis of rotation

$$a = \frac{{}^{S}\omega_{SN}(0)}{|{}^{S}\omega_{SN}(0)|} \tag{44}$$

and, finally, the Euler's angle is

$$\phi(t) = |{}^{S}\omega_{SN}(0)|t + \frac{U}{2}t^2. \tag{45}$$

Interestingly, $R_{BN}(t)$ (correctly found through Eq. (38)) cannot be directly obtained by using Eq. (4), with ${}^{B}\omega_{BN}(t)$ from Eq. (35), as the commutation condition is not verified in this case. □

5.2. *Case of constant torque along the transversal component of the angular momentum vector*

The result introduced by the following theorem regards the case of applied constant torque along the transversal vectorial component of the angular momentum vector of the axially symmetric body, i.e. the component on the plane perpendicular to the symmetry axis $\underline{e}$. The torque direction is also equal to the direction of the transversal component of the angular velocity of the *Virtual Sphere*, because of Eq. (29), and also to the direction of the transversal component of the angular velocity of the body, because of Eq. (20). The development in this section will be exploited later in this paper for the analysis of the nutation canceling maneuver of an axially symmetric spacecraft.

Theorem 3. *Given p_0, q_0, r_0, and U real numbers, the solution of the dynamic problem of determining, at any time t, the absolute angular velocity of a rigid body having two equal principal moments of inertia about the principal body axes σ_1 and σ_2, initial angular velocity components $p(0)=p_0$, $q(0)=q_0$, and $r(0)=r_0$ along the three principal axes, and subjected to an external torque of constant magnitude (m) and always parallel to the transversal vectorial component of the angular momentum vector of the axially symmetric body (and, therefore, to the transversal component of the angular velocity of the Virtual Sphere), is the following:*

$$ {}^{B}\omega_{BN}(t) = \left\{\begin{matrix} \left\{\begin{matrix} p_0\cos(f(t)) + q_0\sin(f(t)) \\ -p_0\sin(f(t)) + q_0\cos(f(t)) \end{matrix}\right\}\left[1+\dfrac{Ut}{\sqrt{p_0^2+q_0^2}}\right] \\ r_0 \end{matrix}\right\}, \tag{46}$$

where $U = m/I$, and

$$f(t) = \left(\frac{I-I_3}{I}\right)r_0 t. \tag{47}$$

Furthermore, the solution of the correspondent kinematic problem of determining, at any time t, the orientation of the body with respect to the inertial frame, i.e. of the coordinate system B with respect to N, is given by

$$R_{BN}(t) = R_{BS}(t)R_{SN}(t), \tag{48}$$

where $R_{BS}(t)$ is given by Eq. (31) with $f(t)$ as in Eq. (47), and $R_{SN}(t)$ is obtained from Eq. (15) of the Corollary 2 of Theorem 1, by substituting the subindex F by S and the time-independent matrix R_{FB} by the following matrix, computed through Eq. (16),

$$R_{FB} = \begin{bmatrix} 0 & \dfrac{q_0}{\sqrt{p_0^2+q_0^2}} & \dfrac{p_0}{\sqrt{p_0^2+q_0^2}} \\ 0 & -\dfrac{p_0}{\sqrt{p_0^2+q_0^2}} & \dfrac{q_0}{\sqrt{p_0^2+q_0^2}} \\ 1 & 0 & 0 \end{bmatrix}. \tag{49}$$

Proof. This theorem follows directly from the analytical developments of Section 4. In particular the angular velocity of the *Virtual Sphere* is given by

$${}^{S}\omega_{SN}(t) = \left\{\begin{matrix} \overline{p}(t) \\ \overline{q}(t) \\ \overline{r}(t) \end{matrix}\right\} = \left\{\begin{matrix} \left\{\begin{matrix} p_0 \\ q_0 \end{matrix}\right\}\left[1+\dfrac{Ut}{\sqrt{p_0^2+q_0^2}}\right] \\ \dfrac{I_3}{I}r_0 \end{matrix}\right\}, \tag{50}$$

as obtained by integrating Eq. (28), with initial conditions given by Eq. (29) and

$$\frac{1}{I}R_{SB}{}^{B}m = \frac{{}^{S}m}{I} = \left\{\begin{matrix} \left\{\begin{matrix} p_0 \\ q_0 \end{matrix}\right\}\left[\dfrac{U}{\sqrt{p_0^2+q_0^2}}\right] \\ 0 \end{matrix}\right\}, \tag{51}$$

which corresponds to the theorem's hypothesis of having a constant external torque $\underline{m}$ parallel to the transversal vectorial component of the angular velocity of the sphere.

Moreover, Eq. (46) is found through the use of Eq. (25), with ${}^{S}\omega_{SN}(t)$ given by Eq. (50), and ${}^{B}\omega_{BS}$ given by Eq. (33) with

$$\omega_e = A\overline{r}(t) = \left(\frac{I-I_3}{I}\right)r_0. \tag{52}$$

Notably, in this case, neither R_{BN} nor R_{SN} can be found through Eq. (4), as the commutation condition is not satisfied in these cases. □

6. Controlling the rotation of an axially symmetric spacecraft by the *virtual sphere* approach

This section introduces two new maneuvers to control the rotation of an axially symmetric spacecraft, namely a *detumbling maneuver* and a *nutation canceling maneuver*. Building upon the results presented in Section 5, it is shown here that the two proposed maneuvers have a complete exact analytic reduction, for the determination of both the absolute angular velocity and the rotation matrix giving the attitude of the spacecraft with respect to the inertial frame.

For both maneuvers we assume that the initial absolute attitude and angular velocity (i.e. $R_{BN}(0)$ and ${}^{B}\omega_{BN}(0)$) are known. Furthermore, without loss of generality, we consider the Assumptions 1 to be valid. Therefore, $R_{BN}(0) = R_{SN}(0)$ and the initial angular velocity of the *Virtual Sphere* (${}^{S}\omega_{SN}(0)$) is given by Eq. (29).

In particular, for the detumbling maneuver, a torque is applied parallel to the angular momentum vector; on the other hand, for the nutation canceling maneuver, a torque is applied parallel to the transversal vectorial component of the angular momentum of the spacecraft.

In practical astronautical applications, for both maneuver cases, the torque can be generated by on-board actuators (e.g. thrusters).

6.1. *Detumbling maneuver*

The proposed detumbling maneuver consists in applying to the axially symmetric spacecraft a torque $\underline{m}$ which is constant in magnitude and kept fixed with the *Virtual Sphere*, in the direction opposite to the angular momentum, and therefore also to the initial angular velocity of the *Virtual Sphere* (see Fig. 2).

The applied torque is therefore expressed in the coordinate system S as

$$ {}^{S}m = m\frac{{}^{S}\omega_{SN}(0)}{|{}^{S}\omega_{SN}(0)|}, \tag{53} $$

where m is a negative constant, equal in magnitude to the value of the acting torque.

The torque is applied for a time duration such that the angular velocity of the *Virtual Sphere* is zeroed. By zeroing the angular velocity of the *Virtual Sphere* also the angular velocity of the axially symmetric spacecraft is zeroed, as it follows from the Reduction Theorem (see Eqs. (22) and (23)).

The proposed detumbling maneuver constitutes a particular case of the motion analyzed in Theorem 2. Therefore the exact analytic solution introduced by Eqs. (35) and (38) (for the dynamics and the kinematics respectively) hold true. In particular, the proposed detumbling maneuver is conducted by applying the following external torque, e.g. with thrusters, to the axially symmetric spacecraft

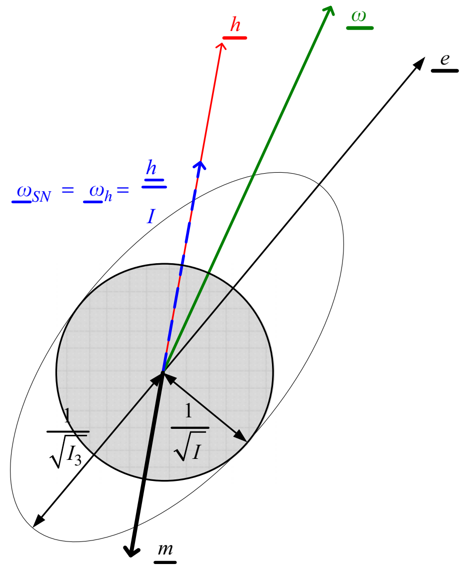

Fig. 2. Conceptual sketch of the proposed detumbling maneuver for an axially symmetric spacecraft, represented by its inertia ellipsoid. The gray area represents the *Virtual Sphere*. The applied torque $\underline{m}$ has direction opposite to the angular momentum vector. The torque is stationary w.r.t. the inertial frame and to the *Virtual Sphere*, while it is moving w.r.t. the spacecraft body.

$$ {}^{B}m(t) = R_{BS}(t){}^{S}m, \tag{54} $$

where $R_{BS}(t)$ is given by Eq. (31) with $f(t)$ as in Eq. (36) and ${}^{S}m$ is given by Eq. (53).

By considering Eq. (35) with $U = m/I$, the maneuver duration Δt, i.e. the time of application of the constant torque in order to zeroing the angular velocity, yields

$$ \Delta t = -\frac{|{}^{S}\omega_{SN}(0)|I}{m}, \tag{55} $$

which is a positive constant, since m is negative.

Notably, the proposed detumbling maneuver is also the optimal detumbling maneuver of minimum time if the maximum available torque is used (in case of spherically constrained maximum torque). Indeed, in this case, the problem is equivalent to the linear one-dimensional problem of bringing the *Virtual Sphere* to a rest condition starting from an arbitrary initial angular rate, by applying the maximum available torque parallel and opposite to the angular velocity vector.

6.2. *Nutation canceling maneuver*

The proposed nutation canceling maneuver consists in applying to the axially symmetric spacecraft a torque $\underline{m}$ which is constant in magnitude, fixed with the *Virtual Sphere*, and having direction opposite to the vectorial component of the angular momentum on the plane normal to the body symmetry axis $\underline{e}$, and, therefore, also opposite to the transversal vectorial component of the angular velocity of the *Virtual Sphere* (see Fig. 3).

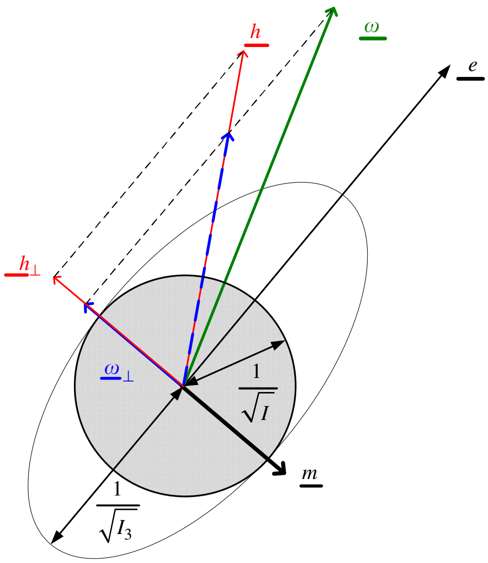

Fig. 3. Conceptual sketch of the proposed nutation canceling maneuver for an axially symmetric spacecraft, represented by its inertia ellipsoid. The gray area in the figure represents the *Virtual Sphere*. The applied torque $\underline{m}$ has direction opposite to the vectorial component of the angular momentum on the plane normal to the axis $\underline{e}$. The torque is stationary with respect to the *Virtual Sphere*, while it is moving with respect to both the inertial frame and the spacecraft body.

The applied torque is therefore expressed in the coordinate system S as

$$^{S}m=m\left\{\begin{array}{c}\left\{\begin{array}{c}p_0\\ q_0\end{array}\right\}\left[\dfrac{1}{\sqrt{p_0^2+q_0^2}}\right]\\ 0\end{array}\right\}, \tag{56}$$

where m is a negative constant.

The torque is applied for a time duration such that the transversal component of the angular velocity of the *Virtual Sphere* is zeroed.

The proposed nutation canceling maneuver constitutes a particular case of the motion analyzed in Theorem 3. Therefore the exact analytic solution introduced by Eqs. (46) and (48) (for the dynamics and kinematics respectively) hold true.

The conceptual rationale for this maneuver is the following: by zeroing the vectorial component of the angular velocity of the *Virtual Sphere* perpendicular to the symmetry axis of the spacecraft (i.e. the axis z of both the coordinate systems S and B), also the transversal component of the angular momentum of the axially symmetric spacecraft is zeroed, as it follows from the Reduction Theorem (Eq. (22)). Therefore, the nutation angle between the spacecraft symmetry axis and the angular momentum vector is canceled. Consequently, the precessing motion of the spacecraft symmetry axis about the angular momentum vector stops and the spacecraft is left in a pure spinning motion about its axis of symmetry, with angular velocity

$$^{B}\omega_{BN}(t)=\left\{\begin{array}{c}0\\ 0\\ r_0\end{array}\right\}. \tag{57}$$

In particular, the proposed nutation canceling maneuver is conducted by applying the following external torque, e.g. with thrusters, to the axially symmetric spacecraft

$$^{B}m(t)=R_{BS}(t)\,^{S}m, \tag{58}$$

where $R_{BS}(t)$ is given by Eq. (31) with $f(t)$ as in Eq. (47), and ^{S}m is given by Eq. (56).

By considering Eq. (46) with $U=m/I$, the maneuver duration Δt, i.e. the time of application of the constant torque in order to zeroing the transversal angular velocity, yields

$$\Delta t=-\frac{\sqrt{p_0^2+q_0^2}I}{m}, \tag{59}$$

which is a positive constant, since m is negative.

Notably, the proposed nutation canceling maneuver is also the optimal nutation canceling maneuver of minimum time if the maximum available torque is used (in case of spherically constrained maximum torque). Indeed, in this case, the problem is equivalent to the linear one-dimensional problem of bringing to zero the transversal component of the angular velocity of the *Virtual Sphere* starting from its arbitrary initial value, by applying the maximum available torque parallel and opposite to the angular velocity vector component.

7. Conclusions

For the first time, to the knowledge of the author, the exact analytic solutions, for both the kinematic and the dynamic problems, have been introduced in this paper for the rotational motion of a rigid body having revolution ellipsoid of inertia, in the following two cases

1. External torque constant in magnitude and parallel to the angular momentum vector.
2. External torque constant in magnitude and parallel to the vectorial component of the angular momentum vector on the plane perpendicular to the axis of the body.

The proposed analytic solutions are valid for any length of time and rotation amplitude. The kinematic solutions are presented in terms of the rotation matrix.

In particular, the results of this paper are obtained by building upon the recently found exact analytic solution for the motion of a rigid body with a spherical ellipsoid of inertia, and by decomposing the motion of an axially symmetric rigid body into the combination of the motion of a *Virtual Sphere* with respect to the inertial frame and that of the axially symmetric body with respect to this *Virtual Sphere*.

Based on these analytical developments, two new spacecraft control maneuvers are proposed which have a complete analytic reduction: namely, a detumbling maneuver and a nutation canceling maneuver. In particular, for the detumbling maneuver the applied torque is kept parallel to the initial angular velocity of the virtual sphere; on the other hand, for the nutation cancelation maneuver, the applied torque is kept parallel to the transversal vectorial component of the angular velocity of the spacecraft. In practice, the requested torque can be generated by on-board actuators, as, for instance, body-fixed thrusters commanded through a modulation and a mapping scheme in order to produce an approximately linear duty cycle (see, for instance, [2]).

The analytical results presented in the paper have been verified by numerical experiments with sample numerical values for the geometry of the spacecraft and the magnitude of the acting torque.

The results presented in this paper have both a theoretical importance for their basic mechanics implications and a practical importance for their related astrodynamic applications.

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